Parameter-Dependent Polynomial Control Issue for Nonlinear Time-Varying System

Shao-Hao Jian, Cheung-Chieh Ku, and Wen-Jer Chang*

***Abstract*—In this paper, a Parameter-Dependent Polynomial Fuzzy (PDPF) control issue of the nonlinear time-varying system is discussed via Sum-Of-Square (SOS) technology. Recently, a new type of fuzzy model named PDPF model is proposed by combining Takagi-Sugeno (T-S) fuzzy model, polynomial representation and Linear Parameter Varying (LPV) description. For the stabilization problem of nonlinear time-varying system, the PDPF controller is designed based on Parallel Distributed Compensation (PDC) method. However, stability problem of PDPF model is often more complex and difficult than one of the traditional Takagi-Sugeno fuzzy model since the nonconvex problem caused by the coupling of variables. To avoid the nonconvex term, a parameter-dependent positive definite matrix in Lyapunov function is adopted to derive the stability criterion. Besides, the convex combination is employed to eliminate the restriction of time-varying parameters. Thus, the sufficient conditions are derived into the SOS form and solved by the SOSTOOLS. Finally, the simulation results of the nonlinear time-varying system are provided to verify the PDPF controller design method.**

Index Terms*—*Linear Parameter Varying System, Nonlinear Time-Varying System, Polynomial Fuzzy Model, Sum Of Squares.

I. INTRODUCTION

The fuzzy control is a popular issue for stabilizing the nonlinear system [1-8]. Referring to [1], the stability sufficient condition was transferred into the linear matrix inequality. It is thus the controller designed problem can be solved by the numerical method. In recent years, the semidefinite programming algorithm has grown rapidly and witnessed the wide application. One of the brunches of the semidefinite programming is Sum Of Squares (SOS) [4]. The SOS technology is based on SOS decomposition, which is consisted of multivariate polynomials. Therefore, the typical Takagi-Sugeno (T-S) fuzzy model can be extended to the polynomial form [5]. The convex combination is containing several polynomial fuzzy subsystems and membership function to represent the local behavior of nonlinear systems. Thus, the overall polynomial fuzzy model is obtained by blending those subsystems and the membership function to describe the nonlinear dynamics. Based on the polynomial description, the polynomial nonlinear terms can be retained in the model so that the number of fuzzy rules can be reduced. Therefore, the control problem of nonlinear systems is simplified by using polynomial fuzzy model since the less number of fuzzy rules. Due to the aforementioned merits of polynomial fuzzy model, it has been adopted for the nonlinear system to investigated different issues, e.g., the sliding mode control for stochastic issue [6], reducing the conservatism by the piecewise linear membership functions dependent method [7] and Robust fuzzy observer-based fault-tolerant control via the homogeneous polynomial Lyapunov function approach [8].

For the nonlinear control issue, the Linear Parameter Varying (LPV) system is aim at describing the time-varying behavior in the nonlinear systems [9-15]. The LPV system has the similar structure to T-S fuzzy model since it was built by the convex combination of the linear time-invarying subsystems and the weighting functions. For the controller designed problem of LPV system, the gain scheduled method was introduced [9]. Because of the similar structure between gain-scheduled controller and the LPV system, the stabilization problems of LPV system can be thus discussed by the gain-scheduled controller design [10]. Thus, LPV system has been utilized to modelling the nonlinear system and investigate the control issues in different performances. For instance, the observer-based controller designed for LPV stochastic systems [11], H_∞ observer-based control [12] and pole assignment issue [13]. Similar to the polynomial fuzzy model, the LPV system was extended to polynomial LPV system by parameter-dependent polynomial subsystems and weighting functions. Thus, the SOS technology is a general method to solve the parameter-dependent polynomial conditions. For the applications of polynomial LPV system, several literatures have been proposed to investigate different types of control issue. One has been applied to the missile longitudinal dynamics controller design problem [14] and the investigation on time-varying behavior of turbo engine [15].

Since the structure between T-S fuzzy model and LPV system is quite similar, the combination of two models was proposed for more general representation to nonlinear time-varying system [16-17]. One of the models is the Parameter-Dependent Polynomial Fuzzy (PDPF) model [17]. In order to achieve stability, the parameter-dependent polynomial Lyapunov function is chosen. Referring to [17], if the upper and lower bounds of the time-varying parameters are the opposite sign, the Lyapunov function cannot be guaranteed to be positive. The restriction in the case of time-varying parameters causes the stability analysis method in [17] is not applicable for some nonlinear time-varying systems. Otherwise, the nonconvex term will exist in the derivative of polynomial Lyapunov function. Some method should be adopted to promote the stability analysis. It is thus an interesting issue to overcome the aforementioned limitations.

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Based on the above motivations, the PDPF control issue for the nonlinear time-varying system is discussed in this paper. According to the polynomial fuzzy representation, the polynomial nonlinear term is remained in the consequent parts of the PDPF model. Therefore, the PDPF model can represent the nonlinear time-varying system by several PDPF subsystems and the membership functions. By the LPV modelling approach, some time-varying terms are considered as the time-varying parameters. Different from the method in [17], the time-varying behavior is represented as the form of convex combination. To achieve the stability, the PDPF fuzzy controller is established by PDC method and the parameter-dependent polynomial Lyapunov function is employed. To deal with the nonconvex problem in the derivative of Lyapunov function, the positive definite matrix is chosen to be independent of polynomial. The sufficient conditions are derived into SOS decomposition which can be solved by the third-party toolbox: SOSTOOLS. Due to the convex combination, the limitation of the case of time-varying parameters is erased and the positive definition of Lyapunov function can be also guaranteed. By solving the sufficient conditions, the feasible solution can be obtained to design the PDPF controller. Finally, the simulation of nonlinear time-varying system is utilized to verify the applicability of the proposed controller designed method.

The structure of the paper is shown as follows. The PDPF description and the related definition and lemma is stated in Section Ⅱ. Section Ⅲ presents the proposed controller designed method. Section Ⅳ exhibits the simulation of the nonlinear time-varying system. Finally, some conclusions are provided in Section Ⅴ.

II. System Description and Problem Statements

In this section, the PDPF model will be introduced and the PDC method will also be considered to design the PDPF controller. Firstly, the nonlinear time-varying system is given as follows.

$$\dot{\boldsymbol{x}} = \Upsilon(\boldsymbol{x}(t), \phi(t), u(t)) \tag{1}$$

where $\boldsymbol{x}(t)$ represents the state vector $\boldsymbol{x}(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_N(t)]$, $\phi(t)$ represents the time-varying parameter and $u(t)$ represents the control input.

For brevity, the notation with respect to time t is omitted, e.g., $\boldsymbol{x}$, u, ϕ and β_h are respectively denoted as $\boldsymbol{x}(t)$, $u(t)$, $\phi(t)$ and $\beta_h(t)$. As (1) showing, the nonlinear system has the time-varying behavior. Therefore, the following PDPF model is introduced to represent (1) and can be completely described by the following PDPF model.

Plant Rule *i* :
IF $\tau_1(t)$ is T_{i1} and … and $\tau_\omega(t)$ is $\mathrm{T}_{i\omega}$ THEN

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_i(\boldsymbol{x}, \phi)\hat{\boldsymbol{x}}(\boldsymbol{x}) + \boldsymbol{B}_i(\boldsymbol{x}, \phi)u \tag{2}$$

where $i = 1, 2, \cdots, r$ and r denotes the number of fuzzy rules, the fuzzy set is denoted by $\mathrm{T}_{i\psi}$, $\tau_\psi(t)$ is the premise variable, $\psi = 1, 2, \cdots, \omega$ is the number of premise variables. $\boldsymbol{A}_i(\boldsymbol{x}, \phi)$ are system matrices, $\boldsymbol{B}_i(\boldsymbol{x}, \phi)$ are input matrices, the term $\hat{\boldsymbol{x}}(\boldsymbol{x})$ is consisted of the monomials in $\boldsymbol{x}$ and one can be represented as $\hat{\boldsymbol{x}}(\boldsymbol{x}) = [\hat{\boldsymbol{x}}_1(\boldsymbol{x}) \quad \hat{\boldsymbol{x}}_2(\boldsymbol{x}) \quad \cdots \quad \hat{\boldsymbol{x}}_N(\boldsymbol{x})]$, N is the number of monomial terms in degree ϖ which is nonnegative integer, $N = n$ when $\varpi = 1$ and $\hat{\boldsymbol{x}}(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = 0$ is held.

Then, the well-known PDC method [1] is applied to the PDPF controller design. Via the PDC method, the controller can be designed for each PDPF subsystem model (2) as follows.

Controller Rule i :
IF $\tau_1(t)$ is T_{i1} and … and $\tau_\omega(t)$ is $\mathrm{T}_{i\omega}$ THEN

$$u = \boldsymbol{F}_i(\hat{\boldsymbol{x}}, \phi)\hat{\boldsymbol{x}}(\boldsymbol{x}) \tag{3}$$

Therefore, the following overall PDPF model is obtained by blending all the rules of (2) and (3).

$$\dot{\boldsymbol{x}} = \textstyle\sum_{i=1}^{r}\sum_{j=1}^{r} w_i(\boldsymbol{x})w_j(\boldsymbol{x})\Big(\boldsymbol{A}_i(\boldsymbol{x}, \phi) + \boldsymbol{B}_i(\boldsymbol{x}, \phi)\boldsymbol{F}_j(\boldsymbol{x}, \phi)\Big)\hat{\boldsymbol{x}}(\boldsymbol{x}) \tag{4}$$

where $w_i(\boldsymbol{x})$ is the membership functions.

Referring to [11], the time-varying parameter can be expressed as the convex combination of weighting functions as follows.

$$\dot{\boldsymbol{x}} = \textstyle\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{h=1}^{l} w_i(\boldsymbol{x})w_j(\boldsymbol{x})\beta_h\Big(\boldsymbol{A}_{ih}(\boldsymbol{x}) + \boldsymbol{B}_{ih}(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x}, \phi)\Big)\hat{\boldsymbol{x}}(\boldsymbol{x}) \tag{5}$$

where β_h is the weighting functions.

An important definition is applied in this paper. The proposed stability conditions should be converted into SOS decomposition which can be computed through semidefinite programming. Thus, the definition of SOS is shown below.

Definition 1: **[18]**

Assume $f^{2\kappa}(\boldsymbol{x})$ is polynomial of even number degree and κ is the positive integer number. $X(\boldsymbol{x})$ is the vector composed of monomial in $\boldsymbol{x}$ with degree ζ satisfied $0 < \zeta \le \kappa$. $f^{2\kappa}(\boldsymbol{x})$ is an SOS if and only if a positive semidefinite matrix $\boldsymbol{Z}$ exists such that the following equality holds.

$$f^{2\kappa}(\boldsymbol{x}) = X^T(\boldsymbol{x})ZX(\boldsymbol{x}) \tag{6}$$

Referring to [18], it should be noted that if $f^{2\kappa}(\boldsymbol{x})$ is SOS then $f^{2\kappa}(\boldsymbol{x}) \ge 0$ but the converse may not be held.

Based on the above definition, the proposed controller designed method and the sufficient conditions is derived in next section.

II. The Proposed Controller Designed Method

By the parameter-dependent polynomial Lyapunov function, the stability criterion is developed to find the feasible solutions. Based on the solutions, one can establish the PDC-based fuzzy controller (3) such that the nonlinear time-varying system (1) is asymptotically stable.

Theorem 1

Given the scalars η and $\varepsilon_1 > 0$, polynomials $\varepsilon_2(\boldsymbol{x}) > 0$ and matrix $\boldsymbol{Y}(\boldsymbol{x})$, the closed-loop system (5) is asymptotically stable if there exist polynomial matrices $\boldsymbol{K}_{jm}(\boldsymbol{x})$ and symmetric matrices $\mathbf{P}_m$ and $\mathbf{Q}_m$ such that

$$\nu^{\mathrm{T}}\left(\mathbf{P}_m - \varepsilon_1 \mathbf{I}\right)\nu \text{ is SOS} \tag{7}$$

$$\nu^{\mathrm{T}}\left(\mathbf{P}_m - \mathbf{Q}_m\right)\nu \text{ is SOS} \tag{8}$$

$$-\nu^T\left(\Phi_{ijhm} - \sum_{m=1}^{l}\eta(\boldsymbol{P}_m - \boldsymbol{Q}_m) + \varepsilon_2(\boldsymbol{x})\boldsymbol{I}\right)\nu \text{ is SOS} \tag{9}$$

where $\Phi_{ijhm} = sym\left\{\boldsymbol{Y}(\boldsymbol{x})\left(\boldsymbol{A}_{ih}(\boldsymbol{x})\boldsymbol{P}_m + \boldsymbol{B}_{ih}(\boldsymbol{x})\boldsymbol{K}_{jm}(\boldsymbol{x})\right)\right\}$, $sym\{\boldsymbol{\eta}\}$ denotes the shorthand notation for $\boldsymbol{\eta} + \boldsymbol{\eta}^{\mathrm{T}}$ and ν is a vector independent of x.

Proof:

Firstly, the candidate of parameter-dependent polynomial Lyapunov function is considered as follows.

$$V(\boldsymbol{x},\phi) = \hat{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{P}^{-1}(\phi)\hat{\boldsymbol{x}}(\boldsymbol{x})$$

It should be noted that the matrix $\boldsymbol{P}^{-1}(\phi)$ is chosen to be independent of $\hat{\boldsymbol{x}}(\boldsymbol{x})$ so that the derivative of $\boldsymbol{P}^{-1}(\phi)$ with respect to x does not require to considered. Next, the time derivative of $V(\boldsymbol{x},\phi)$ is inferred as follows.

$$\dot{V}(\boldsymbol{x},\phi) = \hat{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{P}^{-1}(\phi)\dot{\hat{\boldsymbol{x}}}(\boldsymbol{x}) + \dot{\hat{\boldsymbol{x}}}^T(\boldsymbol{x})\boldsymbol{P}^{-1}(\phi)\hat{\boldsymbol{x}}(\boldsymbol{x}) + \hat{\boldsymbol{x}}^T(\boldsymbol{x})\dot{\boldsymbol{P}}^{-1}(\phi)\hat{\boldsymbol{x}}(\boldsymbol{x}) \tag{10}$$

where $\boldsymbol{Y}(\boldsymbol{x})$ is a polynomial matrix which is the relation between $\dot{\hat{\boldsymbol{x}}}(\boldsymbol{x})$ and $\dot{\boldsymbol{x}}$. One can be represented by $\dot{\hat{\boldsymbol{x}}}(\boldsymbol{x}) = \boldsymbol{Y}(\boldsymbol{x})\dot{\boldsymbol{x}}$ and the (p,q)th elements in $\boldsymbol{Y}(\boldsymbol{x})$ can be calculated by $\boldsymbol{Y}_{pq}(\boldsymbol{x}) = \frac{\hat{x}_p(x)}{x_q}$. Thus, one has the following equality from (10).

$$\dot{V}(\boldsymbol{x},\phi) = \hat{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{P}^{-1}(\phi)\boldsymbol{Y}(\boldsymbol{x})\dot{\boldsymbol{x}} + \boldsymbol{Y}(\boldsymbol{x})^T\boldsymbol{x}^T\boldsymbol{P}^{-1}(\phi)\hat{\boldsymbol{x}}(\boldsymbol{x}) + \hat{\boldsymbol{x}}^T(\boldsymbol{x})\dot{\boldsymbol{P}}^{-1}(\phi)\hat{\boldsymbol{x}}(\boldsymbol{x}) \tag{11}$$

Substituting (5) into (11), one can obtained the following equality.

$$\dot{V}(\boldsymbol{x},\phi) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{h=1}^{l} w_i(\boldsymbol{x})w_j(\boldsymbol{x})\beta_h\hat{\boldsymbol{x}}^T(\boldsymbol{x})\left(\Xi_{ijh} + \dot{\boldsymbol{P}}^{-1}(\phi)\right)\hat{\boldsymbol{x}}(\boldsymbol{x}) \tag{12}$$

where $\Xi_{ijh} = sym\{\boldsymbol{P}^{-1}(\phi)\boldsymbol{Y}(\boldsymbol{x})\left(\boldsymbol{A}_{ih}(\boldsymbol{x}) + \boldsymbol{B}_{ih}(\boldsymbol{x})\boldsymbol{F}_i(\boldsymbol{x},\phi)\right)\}$.

Obviously, if the $\Xi_{ijh} + \dot{P}^{-1}(\phi) < 0$ can be achieved, then $\dot{V}(\boldsymbol{x},\phi) < 0$ is also satisfied which also means the stability is achieved. Multiplying $\mathbf{P}(\phi)$ on both sides of $\Xi_{ijh} + \dot{P}^{-1}(\phi)$, one can obtained the following equality by Lemma 1.

$$P(\phi)\left(\Xi_{ijh} + \dot{P}^{-1}(\phi)\right)P(\phi) = \Psi_{ijh} - \dot{P}(\phi) \tag{13}$$

where

$\Psi_{ijh} = sym\{\boldsymbol{Y}(\boldsymbol{x})\left(\boldsymbol{A}_{ih}(\boldsymbol{x})\boldsymbol{P}(\phi) + \boldsymbol{B}_{ih}(\boldsymbol{x})\boldsymbol{F}_i(\boldsymbol{x},\phi)\boldsymbol{P}(\phi)\right)\}$.

Due to the convex combination, the Lyapunov function can be constructed by the following equalities.

$$\mathbf{P}(\phi) = \sum_{m=1}^{l}\beta_m\mathbf{P}_m \tag{14}$$

$$\dot{\mathbf{P}}(\phi) = \sum_{m=1}^{l}\dot{\beta}_m\mathbf{P}_m \tag{15}$$

$$\boldsymbol{F}_i(\boldsymbol{x},\phi)\boldsymbol{P}(\phi) = \sum_{m=1}^{l}\beta_m\boldsymbol{K}_{jm}(\boldsymbol{x}) \tag{16}$$

Substituting (14), (15) and (16) into (13), one can obtain

$$\Psi_{ijh} - \dot{P}(\phi) = \sum_{m=1}^{l}\beta_m\Phi_{ijhm} - \sum_{m=1}^{l}\dot{\beta}_m P_m \tag{17}$$

Since $\sum_{m=1}^{l}\dot{\beta}_m = 0$, $\sum_{m=1}^{l}\dot{\beta}_m\mathbf{Q}_m = 0$ can be obtained by arbitrary symmetric matrices $\mathbf{Q}_m$. Therefore, one has the following equality.

$$\sum_{m=1}^{l}\dot{\beta}_m\mathbf{P}_m = \sum_{m=1}^{l}\dot{\beta}_m\left(\mathbf{P}_m - \mathbf{Q}_m\right) \tag{18}$$

By setting $\min\{\dot{\beta}_m\} \ge \eta$, one can obtain the following inequality with condition (8).

$$-\sum_{m=1}^{l}\dot{\beta}_m\left(\mathbf{P}_m - \mathbf{Q}_m\right) \le -\sum_{m=1}^{l}\eta\left(\mathbf{P}_m - \mathbf{Q}_m\right) \tag{19}$$

Therefore, the following inequality can be inferred from (12), (13), (17) and (19).

$$\Xi_{ijh} + \dot{P}^{-1}(\phi) \le \Phi_{ijhm} - \sum_{m=1}^{l}\eta(P_m - Q_m) \tag{20}$$

Thus, if (9) holds, one can obtain $\Phi_{ijhm} - \sum_{m=1}^{l}\eta(P_m - Q_m) < 0$ which implies $\dot{V}(\boldsymbol{x},\phi) < 0$ from (20). Thus, if SOS conditions (7-9) holds, the closed-loop system (5) can be guaranteed to achieve asymptotically stable. The proof of this theorem is complete.

In next section, the simulation result is presented to verify the applicability of PDPF model

III. SIMULATION

In this section, the PDPF controller will be designed and applied to the nonlinear time-varying system to verify the applicability of the proposed method. Firstly, a numerical nonlinear time-varying system is considered as follows.

$$\dot{x}_1 = x_1^3 + x_1^2x_2 - x_1x_2^2 + x_1^2 + \left(0.1\sin(t) - 1\right)x_1 \tag{21a}$$

$$\dot{x}_2 = -\sin(x_1) - x_2 \tag{21b}$$

$$\dot{x}_3 = x_1x_3^2 - x_3 \tag{21c}$$

$$\dot{x}_4 = x_1x_2 - x_4 \tag{21d}$$

Based on the PDPF modelling method, one can build the following PDPF model.

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{h=1}^{l} w_i(x_1)w_j(x_1)\beta_h\left(\boldsymbol{A}_{ih}(\boldsymbol{x}) + \boldsymbol{B}_{ih}(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x},\phi)\right)\hat{\boldsymbol{x}}(\boldsymbol{x}) \tag{22}$$

where $\boldsymbol{A}_{11}(\boldsymbol{x}) = \begin{bmatrix} a(\boldsymbol{x})+0.1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ x_3^2 & 0 & -1 & 0 \\ 0 & x_1 & 0 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$$\boldsymbol{A}_{12}(\boldsymbol{x}) = \begin{bmatrix} a(\boldsymbol{x})-0.1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ x_3^2 & 0 & -1 & 0 \\ 0 & x_1 & 0 & -1 \end{bmatrix},$$

$$\boldsymbol{A}_{21}(\boldsymbol{x}) = \begin{bmatrix} a(\boldsymbol{x})+0.1 & 1 & 0 & 0 \\ 0.2172 & -1 & 0 & 0 \\ x_3^2 & 0 & -1 & 0 \\ 0 & x_1 & 0 & -1 \end{bmatrix},$$

$$\boldsymbol{A}_{21}(\boldsymbol{x}) = \begin{bmatrix} a(\boldsymbol{x})-0.1 & 1 & 0 & 0 \\ 0.2172 & -1 & 0 & 0 \\ x_3^2 & 0 & -1 & 0 \\ 0 & x_1 & 0 & -1 \end{bmatrix}, \quad \beta_1 = \frac{1+\sin(t)}{2},$$

$\beta_2 = 1-\beta_1$, $w_1(x_1) = \dfrac{\sin(x_1)+0.2172x_1}{1.2172x_1}$, $w_2(x_1) = 1-w_1(x_1)$

and $a(\boldsymbol{x}) = x_1^2 - x_2^2 + x_1x_2 + x_1 - 1$. Also, the membership function can be referred to Fig. 1.

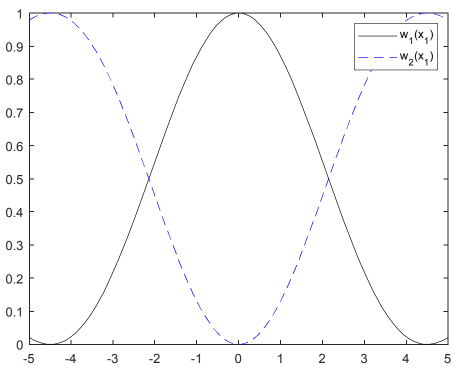

Fig. 1. Membership function of x_1

As the system matrices showing, the upper and lower bound of the time-varying parameter is $-1 \le \phi \le 1$. By applying proposed controller designed method, the parameters are given as $\eta = -0.51$ with $\min\{\dot{\beta}_p\} = -0.5$, $\boldsymbol{Y}(\boldsymbol{x}) = \boldsymbol{I}$, $\varepsilon_1 = 10^{-11}$ and $\varepsilon_2(\boldsymbol{x}) = (x_1^2 + x_2^2 + x_3^2 + x_4^2 + 1) \times 10^{-11}$. Moreover, the maximum degree of $\boldsymbol{K}_{jm}(\boldsymbol{x})$ is set to be two and the degree of $\mathbf{P}_m$ is set to be zero. Using SOSTOOL to Theorem 1 with the given scalars, one can find the feasible solutions by solving the sufficient conditions in Theorem 1 and the feasible solutions are stated in Appendix. Therefore, the following PDPF controller is designed.

$$\boldsymbol{F}_j(\boldsymbol{x}, \phi) = \sum_{m=1}^{2} \beta_m \boldsymbol{K}_{jm}(\boldsymbol{x}) \boldsymbol{P}^{-1}(\phi) \tag{23}$$

Thus, the PDPF controller (3) can be designed by those solutions. After obtaining the PDPF controller, the responses of (21) are shown in Fig. 2 and the initial conditions in four states are set to be $[2 \ \ -2 \ \ -3 \ \ -1]^{\mathrm{T}}$.

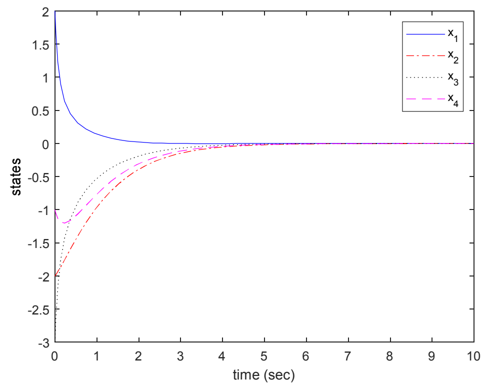

Fig. 2. System responses of states

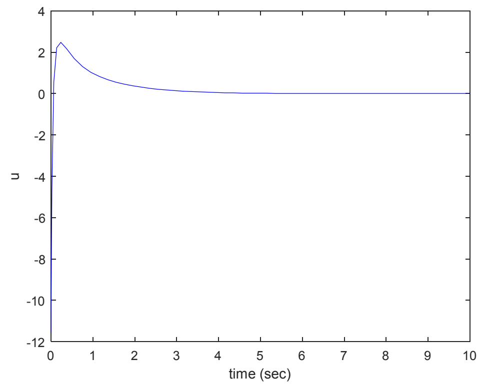

Fig. 3. Responses of controller

Referring to Fig. 2, all states are converged to zero even though the system (21) exists time-varying parameter. Therefore, the applicability and effectiveness of the proposed design method can be furtherly demonstrated by achieving the asymptotical stability of nonlinear time-varying systems

IV. Conclusions

In this paper, the PDPF controller design method was proposed for describing the nonlinear time-varying system and guaranteeing the asymptotical stability. According to the concept of T-S fuzzy model and LPV system, the PDPF model was built by combining the polynomial fuzzy model and convex combination of the time-varying parameter. Through introducing the parameter-dependent Lyapunov function, the SOS conditions were derived to achieve the stability. Based on the LPV description, the restriction on time-varying parameters can be avoided so that the proposed method can be applied to more general nonlinear time-varying systems. Furthermore, the positive definite matrix is chosen to be only dependent on time-varying parameter to avoid the nonconvex problem. Therefore, the designed PDPF controller can achieve best performance. Finally, the simulation results were provided to demonstration the applicability and effectiveness of the proposed controller designed method.

APPENDIX

$$\mathbf{P}_1 = \begin{bmatrix} 0.4788 & -2.594\times10^{-6} & -0.324 & -3.146\times10^{-6} \\ * & 0.8406 & 0.03296 & 0.2922 \\ * & * & 1.016 & 0.0235 \\ * & * & * & 1.219 \end{bmatrix},$$

$$\mathbf{P}_2 = \begin{bmatrix} 0.4788 & -2.963\times10^{-6} & -0.318 & -3.174\times10^{-6} \\ * & 0.8405 & 0.03104 & 0.2921 \\ * & * & 1.015 & 0.0268 \\ * & * & * & 1.219 \end{bmatrix} \text{ and}$$

$$\boldsymbol{K}_{jm}(\boldsymbol{x}) = \begin{bmatrix} k_{jm}^1 & k_{jm}^2 & k_{jm}^3 & k_{jm}^4 \end{bmatrix}$$

where $*$ denotes the transposed elements of matrices for symmetric position, $k_{11}^1 = -0.9437x_1^2 - 0.4481x_1x_2 - 0.4352x_1$

$-0.01827x_2^2 - 4.181\times10^{-9}x_2x_3 - 3.77\times10^{-8}x_2x_4 + 0.007716x_2$
$-1.596\times10^{-8}x_1x_3 - 9.166\times10^{-8}x_1x_4 - 0.4849x_4^2 + 7.016\times10^{-9}x_4$
$-0.4852x_3^2 - 7.558\times10^{-8}x_3x_4 - 4.946\times10^{-9}x_3 - 0.4728$,

$k_{11}^2 = 2.045\times10^{-6}x_1^2 + 2.311\times10^{-6}x_1x_2 + 5.676\times10^{-14}x_1x_3$
$-2.537\times10^{-6}x_2^2 + 1.528\times10^{-14}x_2x_3 + 2.942\times10^{-8}x_2x_4$
$+7.231\times10^{-8}x_3^2 + 5.85\times10^{-14}x_3x_4 - 3.035\times10^{-8}x_3$
$-3.502\times10^{-13}x_1x_4 - 1.058\times10^{-17}x_4^2 - 1.687\times10^{-8}x_4$
$+0.01529x_1 + 0.006254x_2 - 0.3964$,

$k_{11}^3 = 0.324x_1^2 + 0.324x_1x_2 + 3.379\times10^{-9}x_1x_3 + 0.2991x_1$
$-0.324x_2^2 + 6.05\times10^{-9}x_2x_3 - 6.543\times10^{-14}x_2x_4 - 0.001397x_2$
$+8.957\times10^{-8}x_1x_4 - 1.021\times10^{-16}x_4^2 + 3.093\times10^{-8}x_4 - 0.5118$
$-0.4614x_3^2 + 1.7\times10^{-8}x_3x_4 + 1.723\times10^{-8}x_3$,

$k_{11}^4 = 2.985\times10^{-6}x_1^2 + 2.966\times10^{-6}x_1x_2 + 5.093\times10^{-15}x_1x_3$
$-3.072\times10^{-6}x_2^2 + 2.008\times10^{-14}x_2x_3 - 1.603\times10^{-13}x_2x_4$
$+4.864\times10^{-8}x_3^2 + 7.091\times10^{-14}x_3x_4 - 3.606\times10^{-8}x_3$
$+9.484\times10^{-14}x_1x_4 - 1.11\times10^{-17}x_4^2 + 2.708\times10^{-8}x_4$
$+0.00111x_1 + 0.002038x_2 - 0.2594$,

$k_{12}^1 = -0.9348x_1^2 - 0.4393x_1x_2 + 1.053\times10^{-8}x_1x_3 - 0.4262x_1$
$-0.02797x_2^2 + 1.754\times10^{-9}x_2x_3 - 3.275\times10^{-8}x_2x_4 + 0.007509x_2$
$-7.401\times10^{-9}x_1x_4 - 0.4849x_4^2 + 1.498\times10^{-9}x_4 - 0.3923$
$-0.4852x_3^2 + 4.241\times10^{-8}x_3x_4 - 7.215\times10^{-8}x_3$,

$k_{12}^2 = 2.38\times10^{-6}x_1^2 + 2.649\times10^{-6}x_1x_2 - 2.361\times10^{-13}x_1x_3$
$-2.893\times10^{-6}x_2^2 + 8.578\times10^{-14}x_2x_3 - 1.127\times10^{-13}x_2x_4$
$+7.777\times10^{-8}x_3^2 + 1.431\times10^{-13}x_3x_4 - 5.119\times10^{-8}x_3$
$-2.918\times10^{-13}x_1x_4 + 4.825\times10^{-17}x_4^2 + 1.535\times10^{-8}x_4$
$+0.01581x_1 + 0.006445x_2 - 0.4079$,

$k_{12}^3 = 0.318x_1^2 + 0.318x_1x_2 - 4.853\times10^{-8}x_1x_3 + 0.2937x_1$
$-0.318x_2^2 + 2.523\times10^{-8}x_2x_3 + 1.593\times10^{-13}x_2x_4 - 0.001389x_2$
$-9.423\times10^{-15}x_1x_4 + 1.493\times10^{-16}x_4^2 - 1.886\times10^{-8}x_4 - 0.5572$
$-0.4522x_3^2 + 3.561\times10^{-8}x_3x_4 + 5.715\times10^{-8}x_3$,

$k_{12}^4 = 3.016\times10^{-6}x_1^2 + 2.996\times10^{-6}x_1x_2 - 2.728\times10^{-15}x_1x_3$
$-3.1\times10^{-6}x_2^2 + 1.124\times10^{-13}x_2x_3 + 5.796\times10^{-13}x_2x_4$
$+5.886\times10^{-8}x_3^2 + 1.567\times10^{-13}x_3x_4 + 4.058\times10^{-8}x_3$
$-1.226\times10^{-13}x_1x_4 + 8.251\times10^{-18}x_4^2 - 2.674\times10^{-9}x_4$
$+0.00111x_1 - 0.002117x_2 - 0.2594$,

$k_{21}^1 = -0.9433x_1^2 - 0.4479x_1x_2 - 5.202\times10^{-8}x_1x_3 - 0.4374x_1$
$-0.01823x_2^2 - 4.064\times10^{-8}x_2x_3 - 5.798\times10^{-9}x_2x_4 - 0.005119x_2$
$+5.939\times10^{-8}x_1x_4 + 9.128\times10^{-17}x_4^2 - 1.65\times10^{-9}x_4 - 0.741$
$+7.228\times10^{-8}x_3^2 + 8.217\times10^{-14}x_3x_4 + 3.78\times10^{-8}x_3$,

$k_{21}^2 = 2.707\times10^{-6}x_1^2 + 2.574\times10^{-6}x_1x_2 - 3.859\times10^{-14}x_1x_3$
$-2.524\times10^{-6}x_2^2 - 3.217\times10^{-15}x_2x_3 + 3.403\times10^{-14}x_2x_4$
$+9.329\times10^{-14}x_1x_4 + 5.059\times10^{-17}x_4^2 - 4.633\times10^{-8}x_4$
$+7.228\times10^{-8}x_3^2 + 8.217\times10^{-14}x_3x_4 + 3.78\times10^{-8}x_3$
$-0.01183x_1 - 0.005119x_2 - 0.4973$,

$k_{21}^3 = 0.324x_1^2 + 0.324x_1x_2 - 7.365\times10^{-10}x_1x_3 + 0.2993x_1$
$-0.324x_2^2 + 2.529\times10^{-9}x_2x_3 - 6.35\times10^{-9}x_2x_4 - 0.001372x_2$
$-1.828\times10^{-13}x_1x_4 + 5.059\times10^{-17}x_4^2 - 4.633\times10^{-8}x_4 - 0.4973$
$-0.4614x_3^2 + 2.28\times10^{-8}x_3x_4 - 3.062\times10^{-9}x_3$

$k_{21}^4 = 2.967\times10^{-6}x_1^2 + 2.963\times10^{-6}x_1x_2 + 4.714\times10^{-14}x_1x_3$
$-3.072\times10^{-6}x_2^2 + 9.006\times10^{-15}x_2x_3 + 6.25\times10^{-14}x_2x_4$
$+4.854\times10^{-8}x_3^2 + 9.641\times10^{-14}x_3x_4 + 2.323\times10^{-8}x_3$
$-1.128\times10^{-13}x_1x_4 - 2.818\times10^{-17}x_4^2 + 2.226\times10^{-8}x_4$
$-0.001167x_1 - 0.004508x_2 - 0.2689$

$k_{22}^1 = -0.9344x_1^2 - 0.4392x_1x_2 - 4.642\times10^{-8}x_1x_3 - 0.4286x_1$
$-0.02794x_2^2 + 1.859\times10^{-9}x_2x_3 + 4.98\times10^{-8}x_2x_4 + 0.006435x_2$
$+1.126\times10^{-8}x_1x_4 - 0.4849x_4^2 + 2.486\times10^{-9}x_4 - 0.4058$
$-0.4852x_3^2 - 2.342\times10^{-8}x_3x_4 + 2.261\times10^{-8}x_3$,

$k_{22}^2 = 3.047\times10^{-6}x_1^2 + 2.913\times10^{-6}x_1x_2 + 5.896\times10^{-14}x_1x_3$
$-2.88\times10^{-6}x_2^2 + 9.637\times10^{-15}x_2x_3 - 1.238\times10^{-15}x_2x_4$
$+4.544\times10^{-14}x_1x_4 + 2.736\times10^{-17}x_4^2 - 3.399\times10^{-9}x_4$
$+7.778\times10^{-8}x_3^2 + 3.37\times10^{-14}x_3x_4 - 2.035\times10^{-9}x_3$
$+0.01226x_1 - 0.005492x_2 - 0.7384$

$k_{22}^3 = 0.318x_1^2 + 0.318x_1x_2 + 1.16\times10^{-8}x_1x_3 + 0.294x_1$
$-0.318x_2^2 - 3.912\times10^{-9}x_2x_3 - 2.253\times10^{-14}x_2x_4 - 0.001332x_2$
$-6.334\times10^{-14}x_1x_4 + 2.579\times10^{-17}x_4^2 + 2.47\times10^{-8}x_4 - 0.5434$
$-0.4522x_3^2 + 7.353\times10^{-9}x_3x_4 + 1.047\times10^{-8}x_3$ and

$k_{22}^4 = 2.997\times10^{-6}x_1^2 + 2.993\times10^{-6}x_1x_2 + 5.752\times10^{-14}x_1x_3$
$-3.1\times10^{-6}x_2^2 - 2.603\times10^{-14}x_2x_3 - 1.223\times10^{-14}x_2x_4$
$+2.095\times10^{-14}x_1x_4 - 2.122\times10^{-18}x_4^2 - 5.371\times10^{-9}x_4$
$+5.89\times10^{-8}x_3^2 + 3.548\times10^{-14}x_3x_4 - 2.19\times10^{-3}x_3$
$-0.0001974x_1 - 0.000478x_2 - 0.2689$.

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