Smooth Second Order Sliding Mode Stabilization of Underactuated Two-Link Manipulators: The Acrobot and Pendubot Examples

Fazal ur Rehman, Ibrahim Shah, and Waseem Abbasi

***Abstract*—This paper investigates robust stabilization of underactuated two-link manipulators. The well-known benchmark robotic mechanisms of the Acrobot and the Pendubot are taken as case studies. These systems belong to a broader class of underactuated mechanical systems which find many important practical applications. To achieve desired control performance and robustness, we propose a unified control design framework based on the application of a smooth second-order sliding mode (SSOSM) control. Furthermore, we design swingup control laws and use the SSOSM control as balancing control to achieve global stabilization in the presences of disturbances. Simulation results verify the effectiveness of the proposed control design approach. The proposed control design framework can be applied to nonlinear systems other than the two-link manipulators.**

***Index Terms*—two-link manipulator, second-order sliding mode control, Acrobot, Pendubot, underactuated systems.**

I. Introduction

THE diverse and complex nature of the two link manipulators have led researchers and scientist in this field to analyze the system on a case-by-case basis. Examples of underactuated mechanical systems (i.e., with less degrees of freedom) are Acrobot, Pendubot, pendulum systems, TORA system and ball and beam system. These systems are highly nonlinear and exhibit a non-zero degree of underactuation. Acrobot (tip robot) and Pendubot (pendulum robot) are the two-link manipulators and represent the broader class of underactuated mechanical systems which are useful in many practical applications. The Acrobot has an actuated elbow (link 1) and a passive shoulder (link 2) while the Pendubot has a passive elbow (link 1) and an actuated shoulder (link 2). Capturing the features of two degrees of freedom planar robotic mechanisms, the Acrobot [1] and the Pendubot [2] are used as nonlinear benchmark systems for research and education in control theory, robotics, mechatronics, and experimentation. Both the systems have attracted the attention of many researchers in the last few decades and are the subject of excellent research works in the literature.

The control objective for the Acrobot and the Pendubot is same: to move the system from the downward stable position (both links down) to the upward unstable position (both links upward), and stabilize it thereafter. In the literature, as for other pendulum systems like the cart-pendulum and the Furuta pendulum, this control objective is achieved with the design of two controls. First, a swingup control is designed to move the system from the downward unstable position close to the upward unstable position. Second, a stabilizing or balancing control is designed to balance the system at the upward unstable position. Hence, control and stabilization of these system is a challenging and active research problem in field of control theory.

The swingup control is well studied by many researchers and the literature is vast. Different control design techniques used to achieve swingup include: without state feedback using oscillatory inputs [3], [4], energy-based [5], [6], energy-based control combined with neural and fuzzy neural network [7], [8], Lyapunov based [9], intelligent control [10], fuzzy control [11], partial feedback linearization [12], adaptive sliding mode [13], virtual holonomic constraints based design [14], and impulse-momentum approach [15]. On the other hand, methods for balancing control are typically limited to the linearization of the dynamics around the equilibrium point and then using linear control techniques like linear quadratic regulator (LQR) and pole placement to design control. This approach is adopted in most works, for example, [15]–[17].

The reason for using linear methods is that underactuation makes difficult the application of standard nonlinear control methods like exact feedback linearization and backstepping for the design of balancing control. However, a balancing control obtained using the above mentioned linear control techniques has limited practical usefulness due to small region of attraction. Further, the control is not robust to uncertainties and external disturbances and hence results in degraded system performance. A balancing control obtained form robust nonlinear control techniques like sliding mode control (SMC) [18], [19] will not suffer from these drawbacks. This is the motivation behind this work.

However, the application of SMC techniques to the Acrobot and the Pendubot requires special consideration combined with some novel approach because of the complexity of the dynamics. First, we transform the dynamics of these systems to normal forms which consist of a nonlinear subsystem and a linear subsystem. Then we design sliding manifold for the nonlinear subsystem known as the *Lagrangian zero dynamics* and apply smooth second-order sliding mode (SSOSM) to enforce

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sliding mode in the manifold along system dynamics such that stability of the overall system is guaranteed. Although the primary focus of this work is the design of SSOSM based balancing controls, for completeness we also design swingup controls to illustrate swingup and balancing. The application of SSOSM which is both smooth and robust for the design of balancing control of the Acrobot and the Pendubot in the presence of disturbances is the novel contribution of this work.

The rest of the article is organized as follows. Section II presents problem formulation. Section III presents the design SSOSM based balancing controllers. Section IV presents the design of swingup controllers. Section V presents simulation results and discussion and finally section VI concludes the paper.

II. Problem Formulation

Figure 1 shows the schematics of the Acrobot and the Pendubot. The simulated physical parameters [1] are shown in Table I. The dynamics of the Acrobot are described by:

$$m_{11}(q_2)\ddot{q}_1 + m_{12}(q_2)\ddot{q}_2 + c_1(q,\dot{q}) + g_1(q_1,q_2) = 0, \tag{1a}$$

$$m_{21}(q_2)\ddot{q}_1 + m_{22}(q_2)\ddot{q}_2 + c_2(q,\dot{q}) + g_2(q_1,q_2) = \tau + d(t). \tag{1b}$$

The dynamics of the Pendubot are described by:

$$m_{11}(q_2)\ddot{q}_1 + m_{12}(q_2)\ddot{q}_2 + c_1(q,\dot{q}) + g_1(q_1,q_2) = \tau + d(t), \tag{2a}$$

$$m_{21}(q_2)\ddot{q}_1 + m_{22}(q_2)\ddot{q}_2 + c_2(q,\dot{q}) + g_2(q_1,q_2) = 0. \tag{2b}$$

In Eqs. (1) and (2) $q = [q_1, q_2]^T$ is the configuration vector; $m_{11}(q_2)$, $m_{12}(q_2)$,$m_{21}(q_2)$, and $m_{22}(q_2)$ are the elements of the positive definite symmetric inertia matrix; $c_1(q,\dot{q})$, and $c_2(q,\dot{q})$ contains Coriolis and centrifugal terms; $g_1(q_1,q_2)$, and $g_2(q_1,q_2)$ contains the gravitational terms, τ is the control input and $d(t)$ is the matched disturbance. Parameters in Eqs. (1) and (2) are same for both the Acrobot and the Pendubot and are given as:

$$\begin{aligned}
m_{11}(q_2) &= m_1\ell_1^2 + m_2(L_1^2+\ell_2^2) + I_1 + I_2 \\
&\quad + 2m_2L_1\ell_2\cos(q_2), \\
m_{12}(q_2) &= m_2\ell_2^2 + I_2 + m_2L_1\ell_2\cos(q_2), \\
m_{21}(q_2) &= m_{12}, \\
m_{22}(q_2) &= m_2\ell_2^2 + I_2, \\
c_1(q_1,\dot{q}_1,q_2,\dot{q}_2) &= -m_2L_1\ell_2\sin(q_2)(2\dot{q}_1\dot{q}_2+\dot{q}_2^2), \\
c_2(q_1,\dot{q}_1,q_2,\dot{q}_2) &= m_2L_1\ell_2\sin(q_2)\dot{q}_1^2, \\
g_1(q_1,q_2) &= -(m_1\ell_1+m_2L_1)g\sin(q_1) \\
&\quad - m_2\ell_2 g\sin(q_1+q_2), \\
g_2(q_1,q_2) &= -m_2\ell_2 g\sin(q_1+q_2).
\end{aligned} \tag{3}$$

Both the Acrobot and the Pendubot have the following four natural equilibrium points:

P1: $(q_1,\dot{q}_1,q_2,\dot{q}_2) = (0,0,0,0)$,
both link 1 and link 2 up; unstable

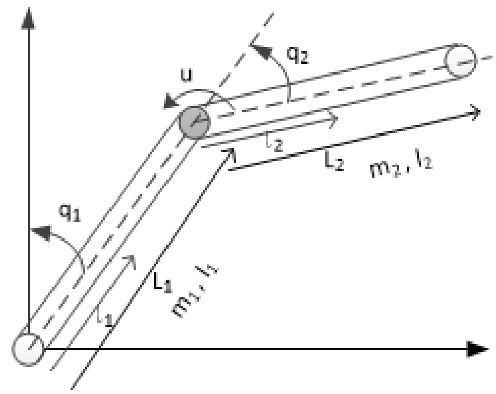
P2: $(q_1,\dot{q}_1,q_2,\dot{q}_2) = (0,0,\pi,0)$,
link 1 up and link 2 down; unstable

P3: $(q_1,\dot{q}_1,q_2,\dot{q}_2) = (\pi,0,\pi,0)$,
link 1 down and link 2 up; unstable

P4: $(q_1,\dot{q}_1,q_2,\dot{q}_2) = (\pi,0,0,0)$,
both link 1 and link 2 down; stable

The control objective is to drive the system from P4 to P1 and balance it there afterward.

The strong nonlinearities in Eq. (3) show the high level of difficulty in control design for the Acrobot and the Pendubot. It is well known that the state space representations of Eq. (1) and Eq. (2) are not directly suitable for control design purposes even in the absence of uncertainties. To facilitate the design of sliding mode control laws for the Acrobot and the Pendubot, we use input and state transformations to put the dynamics in Eqs. (1) and (2) in normal forms.



(a) The Acrobot

Fig. 1: Schematics of the Acrbot and the Pendubot

For the Acrobot, Eq. (1), using the collocated partial feedback linearizing control

$$\tau = (m_{22} - m_{21}m_{11}^{-1}m_{12})w + c_2 + g_2 - m_{21}m_{11}^{-1}(c_1+g_1), \tag{4}$$

where w is a new control input, and the nonlinear coordinate transformation [20]:

$$\begin{aligned}
z_1 &= q_1 + \psi(q_2), \\
z_2 &= m_{11}(q_2)\dot{q}_1 + m_{12}(q_2)\dot{q}_2, \\
\xi_1 &= q_2,\ \xi_2 = \dot{q}_2, \\
\psi(q_2) &= \int_0^{q_2} m_{11}^{-1}(\theta)m_{12}(\theta)d\theta,
\end{aligned} \tag{5}$$

transforms the nominal dynamics of the Acrobot in Eq. (1) into the *normal form*:

$$\ddot{z} = \frac{1}{a+b\cos(\xi)}\Big(k_1\sin(\varphi_1(z,\xi)) + k_2\sin(\varphi_2(z,\xi)) + b\sin(\xi)\dot{\xi}\dot{z}\Big), \tag{6a}$$

$$\ddot{\xi} = w, \tag{6b}$$

where

TABLE I: Simulated physical parameters of the Acrobot and the Pendubot

System	m_1(kg)	m_2(kg)	L_1(m)	L_2(m)	ℓ_1(m)	ℓ_2(m)	I_1(kg.m^2)	I_2(kg.m^2)
Acrobot	1	1	1	2	0.5	1	0.083	0.33
Pendubot	1	1	1	2	0.5	1	0.083	0.33

$$\begin{aligned}
\varphi_1(z,\xi) &= z - \frac{\xi}{2} - w_1 \tan^{-1}\left(w_2 \tan\left(\frac{\xi}{2}\right)\right),\\
\varphi_2(z,\xi) &= z + \frac{\xi}{2} - w_1 \tan^{-1}\left(w_2 \tan\left(\frac{\xi}{2}\right)\right),\\
w_1 &= \frac{2c-a}{\sqrt{a^2-b^2}},\\
w_2 &= \sqrt{\frac{a-b}{a+b}},\\
a &= m_1\ell_1^2 + m_2(L_1^2+\ell_2^2) + I_1 + I_2,\\
b &= 2m_2L_1\ell_2,\\
c &= m_2\ell_2^2 + I_2,\\
k_1 &= (m_1\ell_1 + m_2L_1)g,\\
k_2 &= m_2\ell_2 g.
\end{aligned} \tag{7}$$

For the Pendubot, Eq. (2), using the noncollocated partial feedback linearizing control

$$\tau = (m_{12} - m_{11}m_{21}^{-1}m_{22})w + c_1 + g_1 - m_{11}m_{21}^{-1}(c_2 + g_2), \tag{8}$$

where w is a new control input, and the nonlinear coordinate transformation [20]:

$$\begin{aligned}
z_1 &= q_1 + \psi(q_2),\\
z_2 &= m_{21}(q_2)\dot{q}_1 + m_{22}(q_2)\dot{q}_2,\\
\xi_1 &= q_2,\ \xi_2 = \dot{q}_2, \int_0^{q_2} m_{21}^{-1}(\theta)m_{22}(\theta)d\theta,
\end{aligned} \tag{9}$$

transforms the nominal dynamics of the Pendubot in Eq. (2) into the *normal form*:

$$\ddot{z} = \frac{1}{c+b\cos(\xi)}\Big(k\sin(\varphi(z,\xi)) - b\sin(\xi)\Big(\Big(\dot{z} - \frac{c\dot{\xi}}{c+b\cos(\xi)}\Big)^2 - \frac{c\dot{\xi}^2}{c+b\cos(\xi)}\Big)\Big), \tag{10a}$$

$$\ddot{\xi} = w, \tag{10b}$$

where

$$\begin{aligned}
\varphi(z,\xi) &= z + \xi - w_1 \tan^{-1}\left(w_2 \tan\left(\frac{\xi}{2}\right)\right),\\
w_1 &= \frac{2c}{\sqrt{c^2-b^2}},\\
w_2 &= \sqrt{\frac{c-b}{c+b}},\\
a &= m_1\ell_1^2 + m_2(L_1^2+\ell_2^2) + I_1 + I_2,\\
b &= m_2L_1\ell_2,\\
c &= m_2\ell_2^2 + I_2,\\
k &= m_2\ell_2 g.
\end{aligned} \tag{11}$$

The transformed dynamics (6) and (10) of the Acrobot and the Pendubot are of the following more general:

$$\ddot{z} = f(z,\dot{z},\xi,\dot{\xi}), \tag{12a}$$

$$\ddot{\xi} = w + D(z,\dot{z},\xi,\dot{\xi},t), \tag{12b}$$

where $D(z,\dot{z},\xi,\dot{\xi},t)$ is included to represent the lumped effect of all uncertainties/disturbances after transformation.

In (12) the first block represents the *Lagrangian zero dynamics* for the second block. Treating ξ as control input for the first block, the form reduces the control of the Acrobot (1) and the Pendubot (2) to the control of the reduced order $z-$subsystem in (12a). To stabilize the Acrobot (1) and the Pendubot (2), we stabilize their transformed normal forms in Eqs. (6) and (10), first taking the more general form (12) into consideration.

Stability of the nominal $\xi-$subsystem in (12b), in general and for the Acrobot and Pendubot in Eqs. (6) and (10) in specific, does not imply stability of the Lagrangian zero dynamics in (12a), i.e., the following *second order Lagrangian zero dynamics*

$$\ddot{z} = f(z,\dot{z},0,0), \tag{13}$$

are unstable and hence we conclude that stability of (12b) does not imply stability the overall system (12).

Therefore we analyze stability of the *Lagrangian zero dynamics* in Eq. (12a) with z as output. If $z \equiv 0$ then the zero dynamics are governed by Eq. (12a), with $(z=0, \dot{z}=0, \ddot{z}=0)$, as below:

$$f(0,0,\xi,\dot{\xi}) = 0. \tag{14}$$

If the *first order zero dynamics* in (14) are stable then stabilization of the $z-$subsystem (12a) will render the overall system (12) stable. To make the $z-$subsystem (12a) stable we need the following condition to be satisfied

$$f(z,\dot{z},\xi,\dot{\xi}) = -\alpha\dot{z} - \beta z, \tag{15}$$

with $\alpha > 0$, $\beta > 0$ as design constants. To meet condition (15) we design the sliding manifold as below:

$$\sigma = f(z,\dot{z},\xi,\dot{\xi}) + \alpha\dot{z} + \beta z. \tag{16}$$

When sliding mode is established in (16), condition (15) is satisfied and the dynamics in (12a) become

$$\ddot{z} + \alpha\dot{z} + \beta z = 0, \tag{17}$$

which is a stable linear system for $\alpha > 0$, $\beta > 0$. The stability of $\xi-$subsystem follows from the stability of the zero dynamics in (14) and hence the overall system (12) becomes stable.

Assumption 1. *The origin in the system state space is an equilibrium point of the open loop Lagrangian zero dynamics* *subsystem* (12a) *i.e.,* $f(0,0,0,0) = 0$.

Assumption 2. *The manifold* $f(0,0,\xi,\dot{\xi}) = 0$ *in Eq.* (14) *is stable.*

Assumption 3. *The existence of well defined relative degree and controllability requires* $\dfrac{\partial f}{\partial \xi} \neq 0$ *and* $\dfrac{\partial f}{\partial \dot{\xi}} \neq 0$.

Assumption 4. *The transformed uncertainties* $D(z,\dot{z},\xi,\dot{\xi},t)$ *is bounded as* $\left|D(z,\dot{z},\xi,\dot{\xi},t)\right| \leq D_0$

Remark 1. *Assumption 1 is satisfied for the Acrobot* (6) *and the Pendubot* (10).

Remark 2. *Eq.* (14) *leads to an algebraic equation for the Acrobot* (6) *and hence, keeping in view Assumption 1, Assumption 2 is satisfied for the Acrobot* (6).

Remark 3. *Neglecting higher order terms, Eq.* (14) *leads to an algebraic equation for the Pendubot* (10) *and hence, keeping in view Assumption 1, Assumption 2 is satisfied for the Pendubot* (10).

Remark 4. *Assumption 3 is not satisfied for the Acrobot* (6) *and the Pendubot* (10). *We modify the definition of sliding manifold* σ *in Eq.* (16) *to satisfy Assumption 3.*

III. Smooth Second Order Sliding Mode Control of the Acrobot and Pendubot

This section presents second order sliding mode control of the Acrobot and the Pendubot.

A. Acrobot

The dynamics in (6a) show that, after z converges to zero, the $\xi-$dynamics are governed by the algebraic equation:

$$k_1 \sin(\varphi_1(0,\xi)) + k_2 \sin(\varphi_2(0,\xi)) = 0, \tag{18}$$

and by Assumptions 1 and 2 (Remark 2) the solution of this equation is $\xi = 0$ and hence ξ tends to zero upon the stabilization of $z-$subsystem (6a).

We note that in Eq. (6a), the denominator term $(a + b\cos(\xi))$ is strictly positive for $-\frac{\pi}{2} < \xi < \frac{\pi}{2}$, and hence to achieve the stable system in Eq. (17), we choose the sliding manifold as below:

$$\sigma = k_1 \sin(\varphi_1(z,\xi)) + k_2 \sin(\varphi_2(z,\xi)) + \alpha \dot{z} + \beta z. \tag{19}$$

The last term $b\sin(\xi)\dot{\xi}\dot{z}$ in Eq. (6a) is taken into account in controller synthesis but excluded in the design of sliding manifold in Eq. (19) for the following reasons:

i. including this term in the sliding manifold makes Assumption 3 invalid.
ii. being a third order, is small near the origin.
iii. the coefficient α of $\dot{z}$ in the sliding manifold (19) can be chosen sufficiently large to dominate the state dependent coefficient $b\sin(\xi)\dot{\xi}$ of $\dot{z}$ in this term.

Reasons (ii) and (iii) are crude assumptions but simulation results justify their validity.

Since σ does not depend explicitly on $\dot{\xi}$, the relative degree of system (19) is 2. For the control w to appear, we take twice the time derivative of σ along the dynamics (6) and achieve:

$$\ddot{\sigma} = g(z,\dot{z},\xi,\dot{\xi}) + u, \tag{20}$$

where $g(z,\dot{z},\xi,\dot{\xi})$ is a drift term containing the uncertain term $\dfrac{\partial f}{\partial \xi} D(z,\dot{z},\xi,\dot{\xi},t)$ and

$$u = h(z,\dot{z},\xi,\dot{\xi})w, \tag{21}$$

$$h(z,\dot{z},\xi,\dot{\xi}) = \frac{\partial f}{\partial \xi}. \tag{22}$$

To enforce sliding mode in relative degree 2 system (20) we use the following smooth second order sliding mode control law [21]:

$$u = -s_2 - K_1|\sigma|^{(\rho-2)/\rho}\text{sign}(\sigma) - K_2|\dot{\sigma}|^{(\rho-2)/(\rho-1)}\text{sign}(\dot{\sigma}), \tag{23}$$

where $\rho \geq 2$ and $K_1 > 0$, $K_2 > 0$ are design constants.

The uncertain bounded drift term $g(z, \dot{z}, \xi, \dot{\xi})$ in (20) is estimated with following observer [21] ($m = 2$) as $s_2 = \hat{g}(z,\dot{z},\xi,\dot{\xi})$:

$$\begin{aligned} \dot{s}_0 &= s_1, \\ \dot{s}_1 &= v_1 + u, \\ v_1 &= -\lambda_2|\Lambda|^{1/3}|s_1 - \dot{\sigma}|^{2/3}\text{sign}(s_1 - \dot{\sigma}) + s_2, \dot{s}_2 \end{aligned} \tag{24}$$

where λ_2 and λ_1 are design parameters and $\Lambda > 0$ is Lipshitz constant of $\ddot{g}(z,\dot{z},\xi,\dot{\xi})$. Further the observer also estimate $\dot{\sigma}$ as $s_1 = \hat{\dot{\sigma}}$.

Theorem 1. *The closed loop system* (20), (23), (24) *is finite time stable and hence* σ, $\dot{\sigma}$ *converge to* 0 *in finite time.*

Proof. The proof can be found in [21]. □

In terms of the actual coordinates $(q_1, \dot{q}_1, q_2, \dot{q}_2)$ we have:

$$\begin{aligned} \sigma = k_1 \sin(q_1) + k_2 \sin(q_1 + q_2) + \alpha \dot{q}_1 + \frac{\alpha \dot{q}_2 \,(1 + h_0)}{2} \\ + \beta \left(q_1 + \frac{q_2}{2} + w_1 \tan^{-1}\left(w_2 \tan\left(q_2/2\right)\right)\right), \end{aligned} \tag{25}$$

$$\begin{aligned} \dot{\sigma} = k_1 \dot{q}_1 \cos(q_1) + k_2(\dot{q}_1 + \dot{q}_2)\cos(q_1 + q_2) + \beta \dot{q}_1 \\ + \frac{\beta \dot{q}_2 \,(1 + h_0)}{2} + \alpha h_1, \end{aligned} \tag{26}$$

and $h(z,\dot{z},\xi,\dot{\xi})$ in Eq. (21) is:

$$\begin{aligned} h(z,\dot{z},\xi,\dot{\xi}) = \frac{1}{2}k_1(-1 - h_0)\cos(q_1) + \frac{1}{2}k_2(+1 - h_0) \\ (\cos(q_1 + q_2)) + \frac{\alpha b \left(2\dot{q}_1 + \dot{q}_2 \,(1 + h_0)\right)\sin(q_2)}{2a + 2b\cos(q_2)}, \end{aligned} \tag{27}$$

where

$$h_0 = \frac{w_1 w_2 \sec^2(q_2/2)}{1+w_2^2\tan^2(q_2/2)},$$
$$h_1 = \frac{1}{2a+2b\cos(q_2)}\Big(b\sin(q_2)\dot{q}_2\big(2\dot{q}_1+\dot{q}_2\big(1+h_0\big)\big)h_2 + 2k_1\sin(q_1)+2k_2\sin(q_1+q_2)\Big).$$

The final control τ for the Acrobot (1) is given by (4) with w given by (21) and u given by (23) and (24). Fig. 2 shows simulation results for the Acrobot (1) further discussed in Section V.

B. Pendubot

The dynamics in (10a) show that, after z converges to zero, the $\xi-$dynamics are governed, neglecting the higher order terms, by the algebraic equation:

$$k\sin(\varphi(0,\xi)) = 0, \tag{28}$$

and by Assumptions 1 and 2 (Remark 2) the solution of this equation is $\xi = 0$ and hence ξ tends to zero upon the stabilization of $z-$subsystem (10a).

We note that in Eq. (10a), the denominator term $(c+b\cos(\xi))$ is strictly positive for $-\frac{\pi}{2} < \xi < \frac{\pi}{2}$, and hence to achieve the stable system in Eq. (17), we choose the sliding manifold as below:

$$\sigma = k\sin(\varphi(z,\xi)) + \alpha\dot{z} + \beta z. \tag{29}$$

The last term in Eq. (10a) is taken into account in controller synthesis but excluded in the design of sliding manifold in Eq. (29) for the following reasons:

i. including this term in the sliding manifold makes Assumption 3 invalid.
ii. being higher order, is small near the origin.

Reason (ii) is a crude assumption but simulation results justify its validity. Similar to the Acrobot case we take twice the time derivative of σ along the dynamics (10) and achieve:

$$\ddot{\sigma} = g(z,\dot{z},\xi,\dot{\xi}) + u, \tag{30}$$

where $g(z,\dot{z},\xi,\dot{\xi})$ is a drift term containing the uncertain term $\dfrac{\partial f}{\partial \xi}D(z,\dot{z},\xi,\dot{\xi},t)$ and

$$u = h(z,\dot{z},\xi,\dot{\xi})w, \tag{31}$$

$$h(z,\dot{z},\xi,\dot{\xi}) = \frac{\partial f}{\partial \xi}. \tag{32}$$

To enforce sliding mode in relative degree 2 system (30) we use Theorem 1 with control law (23) and observer (24)

The sliding variable σ in Eq. (29) in terms of the actual coordinates $(q_1,\dot{q}_1,q_2,\dot{q}_2)$ is:

$$\begin{aligned}\sigma = {}& k\sin(q_1+q_2) + \alpha(\dot{q}_1 + h_0\dot{q}_2) \\ & + \beta\left(q_1 + w_1\tan^{-1}\left(w_2\tan\left(q_2/2\right)\right)\right),\end{aligned} \tag{33}$$

$$\dot{\sigma} = k(\dot{q}_1+\dot{q}_2)\cos(q_1+q_2) + \beta(\dot{q}_1+h_0\dot{q}_2) + \alpha h_1, \tag{34}$$

and $h(z,\dot{z},\xi,\dot{\xi})$ in Eq. (31) is:

$$\begin{aligned}h(z,\dot{z},\xi,\dot{\xi}) = {}& k(1-h_0)\cos(q_1+q_2) \\ & + \frac{1}{(a+2b\cos(q_2))^3}\Big(2\alpha bc\sin(q_2)\big(\big(\dot{q}_1+h_0\dot{q}_2\big) \\ & (a+2b\cos(q_2)) - c\dot{q}_2 + (a+2b\cos(q_2))\dot{q}_2\big)\Big),\end{aligned} \tag{35}$$

where

$$h_0 = \frac{1}{2}\Big(\frac{w_1 w_2\sec^2\big(q_2/2\big)}{1+w_2^2\tan^2\big(q_2/2\big)}\Big),$$
$$\begin{aligned}h_1 = {}& \frac{1}{(a+2b\cos(q_2))^3}\Big(k(a+2b\cos(q_2))^2\sin(q_1+q_2) \\ & - b\sin(q_2)\big(\big(\big(\dot{q}_1+h_0\dot{q}_2\big)(a+2b\cos(q_2)) - c\dot{q}_2\big)^2 \\ & - c(a+2b\cos(q_2))\dot{q}_2^2\big)\Big).\end{aligned}$$

The final control τ for the Pendubot (2) is given by (8) with w given by (21) and u given by (23) and (24). Fig. 3 shows simulation results for the Pendubot (2) further discussed in Section V.

IV. Swingup Control of the Acrobot and the Pendubot

For global stabilization, from the downward stable equilibrium position $q_1 = \pi, q_2 = 0$ to the upward unstable equilibrium position $q_1 = 0, q_2 = 0$, we design swingup control laws for the Acrobot and the Pendubot in this section and use the SSOSM control as balancing control.

A. Acrobot

We partially linearize the dynamics of the Acrobot with respect to q_2. Solving Eq. (1a) for $\ddot{q}_1$ we get:

$$\ddot{q}_1 = -m_{11}^{-1}\left(c_1 + g_1 + m_{12}\ddot{q}_2\right). \tag{36}$$

Putting $\ddot{q}_1$ into Eq. (1b) and using the following collocated Partial Feedback Linearizng (PFL) control

$$\begin{aligned}\tau_{PFL2} = {}& (m_{22} - m_{21}m_{11}^{-1}m_{12})w_{swingup} \\ & + c_2 + g_2 - m_{21}m_{11}^{-1}(c_1+g_1),\end{aligned} \tag{37}$$

where $w_{swingup}$ is a new control to be designed, the dynamics of the Acrobot become:

$$m_{11}\ddot{q}_1 + c_1 + g_1 = -m_{12}w_{swingup}, \tag{38a}$$
$$\ddot{q}_2 = \quad w_{swingup}. \tag{38b}$$

To stabilize Eq. (38b) we choose the following control law

$$w_{swingup} = -K_d\dot{q}_2 - K_p q_2, \tag{39}$$

with K_d, K_p as positive design constants. Once q_2 is stabilized, $w_{swingup}$ becomes zero and the q_1-dynamics, in accordance with Eq. (36) (equivalently, Eq. (38a)), become

$$\begin{aligned}&\big(m_1\ell_1^2 + m_2(L_1^2+\ell_2^2) + I_1 + I_2 + 2m_2L_1\ell_2\big)\ddot{q}_1 \\ & - g\big(m_1\ell_1 + m_2L_1 + m_2\ell_2\big)\sin(q_1) = 0.\end{aligned} \tag{40}$$

Fig. 4 shows closed loop response of the Acrobot (1) with $w_{swingup}$ using $K_d = 12.0$, $K_p = 36.0$ for the initial condition

$q(0) = [\pi, 0, 2\pi, 0]^T$. The pendulum behavior in Eq. (40) is shown Fig. 4a.

Fig. 5 shows a successful swingup using $w_{swingup}$ with $K_d = 12.0$, $K_p = 36.0$ and then balancing by the SSOSM controller in the presence of external disturbance for the initial condition $q(0) = [\pi, 0, 2\pi, 0]^T$.

B. Pendubot

We partially linearize the dynamics of the Pendubot with respect to q_1. Solving Eq. (2b) for $\ddot{q}_2$ we get:

$$\ddot{q}_2 = -m_{22}^{-1} \left(c_2 + g_2 + m_{21}\ddot{q}_1\right). \tag{41}$$

Putting $\ddot{q}_2$ the result in Eq. (2a) and using the following collocated Partial Feedback Linearizng (PFL) control

$$\begin{aligned} \tau_{PFL1} = {} & (m_{11} - m_{12}m_{22}^{-1}m_{21})w_{swingup} \\ & + c_1 + g_1 - m_{12}m_{22}^{-1}(c_2 + g_2), \end{aligned} \tag{42}$$

where $w_{swingup}$ is a new control to be designed, the dynamics of the Pendubot become:

$$\ddot{q}_1 = \quad v_{swingup}, \tag{43a}$$

$$m_{22}\ddot{q}_2 + c_2 + g_2 = -m_{21}w_{swingup}. \tag{43b}$$

To stabilize (43a) we choose the following control law

$$w_{swingup} = -K_d\dot{q}_1 - K_p q_1, \tag{44}$$

with K_d, K_p as positive design constants. Once q_1 is stabilized, $w_{swingup}$ becomes zero and the q_2-dynamics, in accordance with Eq. (41) (equivalently, Eq. (43b)), become

$$\left(m_2\ell_2^2 + I_2\right)\ddot{q}_2 - m_2\ell_2 g\sin(q_2) = 0. \tag{45}$$

Fig. 6 shows closed loop response of the Pendubot (2) with $w_{swingup}$ with $K_d = 10.0$, $K_p = 125.0$ for the initial condition $q(0) = [\pi, 0, 0, 0]^T$. The Pendulum behavior in Eq. (45) is shown Fig. 6b.

Fig. 7 shows a successful using $w_{swingup}$ with $K_d = 10.0$, $K_p = 125.0$ and then balancing by the SSOSM controller in the presence of external disturbance for the initial condition $q(0) = [\pi, 0, 0, 0]^T$.

V. Simulation Results and Discussion

This section presents simulation results and discussion for the design examples.

The Acrobot: Fig. 2 shows closed loop response of the Acrobot with SSOSM control u (23) ($\rho = 3$, $K_1 = 75$, $K_2 = 50$), observer (24) ($\lambda_1 = 1$, $\lambda_2 = 3$), sliding parameters ($\alpha = 8$, $\beta = 16$). The controller stabilizes the system from the initial condition $q(0) = \left[-\frac{\pi}{6}, 0, \frac{\pi}{3}, 0\right]^T$ to the upward unstable equilibrium position $q = [0, 0, 0, 0]^T$ in less than 6 seconds. The control effort is smooth. Fig. 4 shows closed loop response with $w_{swingup}$ (39) with parameters $K_d = 12.0$, $K_p = 36.0$. The controller stabilize the q_1-dynamics as desired but the q_2-dynamics behave as in (40). Fig. 5 shows successful swing up, from $q_2 = \pi$ to $q_2 = 0$, with $w_{swingup}$ (39) and then balancing with u (23) in the presence of external disturbance $d(t) = 2\sin(\pi t)$.

The Pendubot: Fig. 3 shows closed loop response of the Pendubot with SSOSM control u (23) ($\rho = 3$, $K_1 = 125$, $K_2 = 100$), observer (24) ($\lambda_1 = 1$, $\lambda_2 = 3$), sliding parameters ($\alpha = 8$, $\beta = 16$). The controller stabilizes the system from the initial condition $q(0) = \left[-\frac{\pi}{6}, 0, \frac{\pi}{3}, 0\right]^T$ to the upward unstable equilibrium position $q = [0, 0, 0, 0]^T$ in less than 4 seconds. The control effort is smooth. Fig. 6 shows closed loop response with $w_{swingup}$ (44) with parameters $K_d = 10.0$, $K_p = 125.0$. The controller stabilize the q_1-dynamics as desired but the q_2-dynamics behave as in (45). Fig. 7 shows successful swing up, from $q_2 = \pi$ to $q_2 = 0$, with $w_{swingup}$ (44) and then balancing with u (23) in the presence of external disturbance $d(t) = 5\sin(\pi t)$.

VI. Conclusion

The application of smooth second order sliding mode control was investigated for two-links manipulator taking the benchmark systems of the Acrobot and the Pendubot as case studies. The enhance control performance and robustness was demonstrated with simulation results. Moreover the control action is smooth. Both smoothness and robustness of control action are highly demanded for mechanical control systems operating under uncertainty conditions. For completeness, swingup controls were designed and successful swingup followed by balancing were demonstrated.

Acknowledgment

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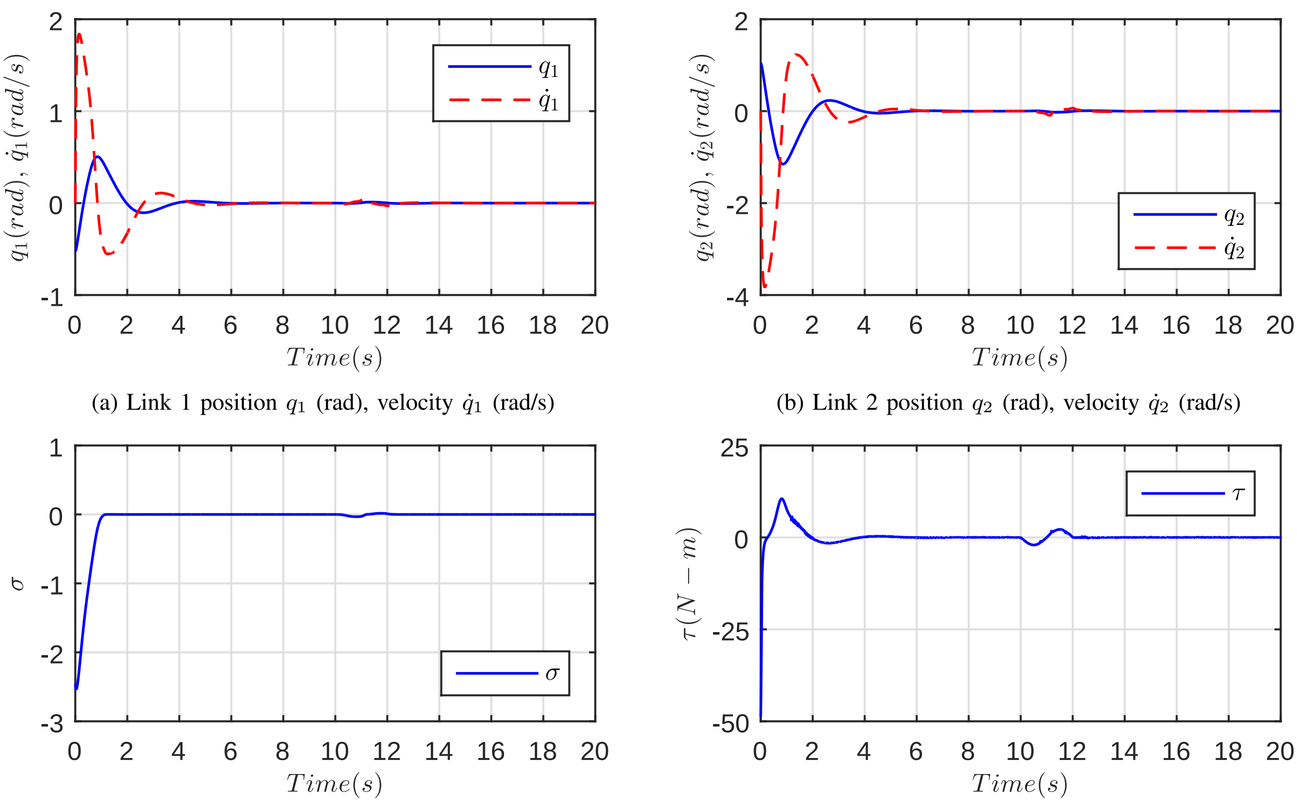


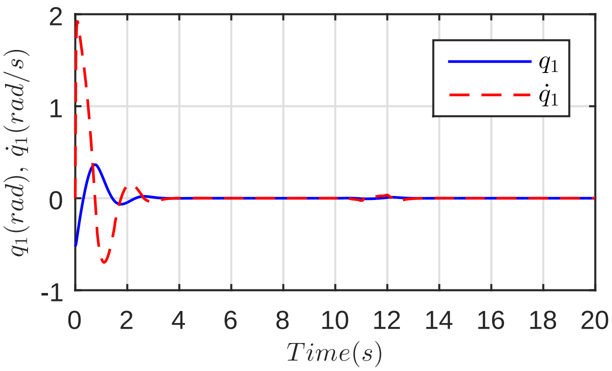
(a) Link 1 position q_1 (rad), velocity $\dot{q}_1$ (rad/s)

(b) Link 2 position q_2 (rad), velocity $\dot{q}_2$ (rad/s)

(c) Sliding surface σ

(d) Control effort τ (N-m)

Fig. 2: Acrobot - Closed loop response with SOSM control u (23) ($\rho = 3$, $K_1 = 75$, $K_2 = 50$), observer (24) ($\lambda_1 = 1$, $\lambda_2 = 3$), sliding parameters ($\alpha = 8$, $\beta = 16$), initial condition $q(0) = \left[-\frac{\pi}{6}, 0, \frac{\pi}{3}, 0\right]^T$, applied disturbance $d(t) = 2\sin(\pi t)$ from $t = 10$ (s) to $t = 12$ (s).



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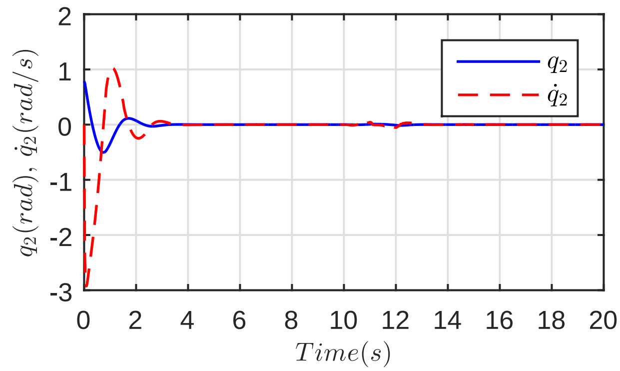

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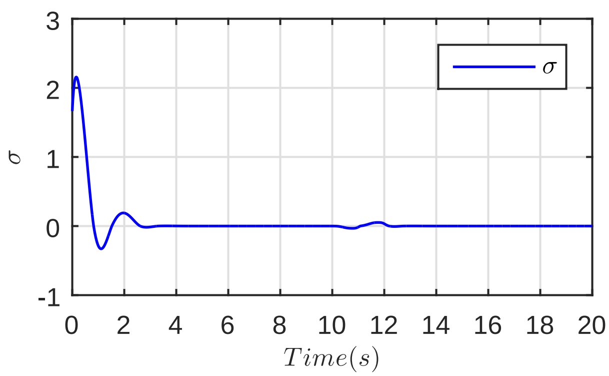
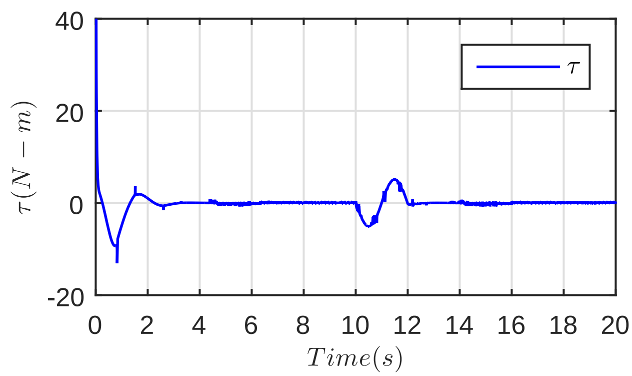
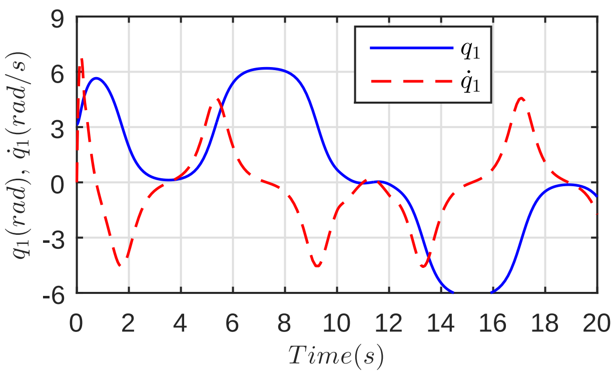


(c) Sliding surface σ

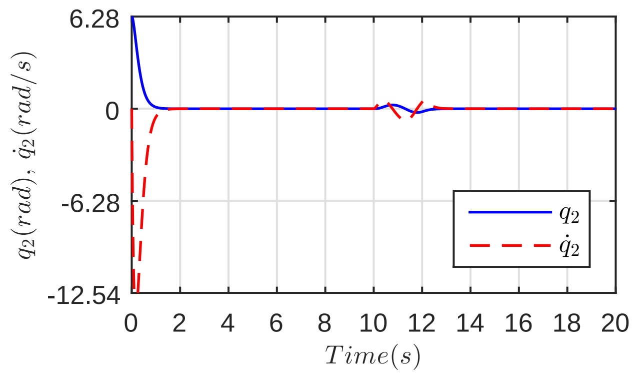


(d) Control effort τ (N-m)

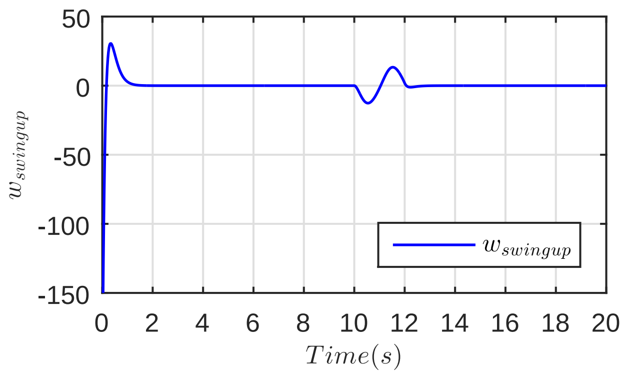
Fig. 3: Pendubot - Closed loop response with SOSM control u (23) ($\rho = 3$, $K_1 = 125$, $K_2 = 100$), observer (24) ($\lambda_1 = 1$, $\lambda_2 = 3$), sliding parameters ($\alpha = 8$, $\beta = 16$), initial condition $q(0) = \left[-\frac{\pi}{6}, 0, \frac{\pi}{3}, 0\right]^T$, applied disturbance $d(t) = 5\sin(\pi t)$ from $t = 10$ (s) to $t = 12$ (s).



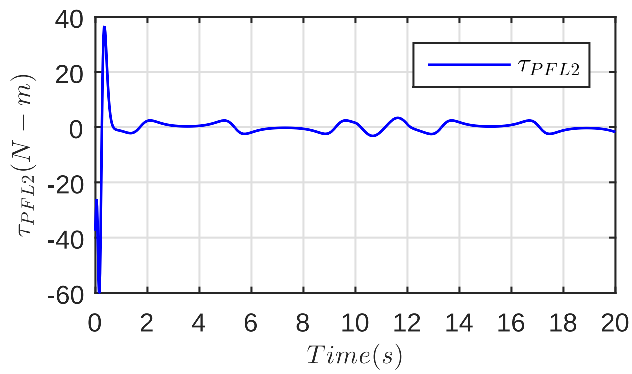
(a) Link 1 position q_1 (rad), velocity $\dot{q}_1$ (rad/s)



(b) Link 2 position q_2 (rad), velocity $\dot{q}_2$ (rad/s)

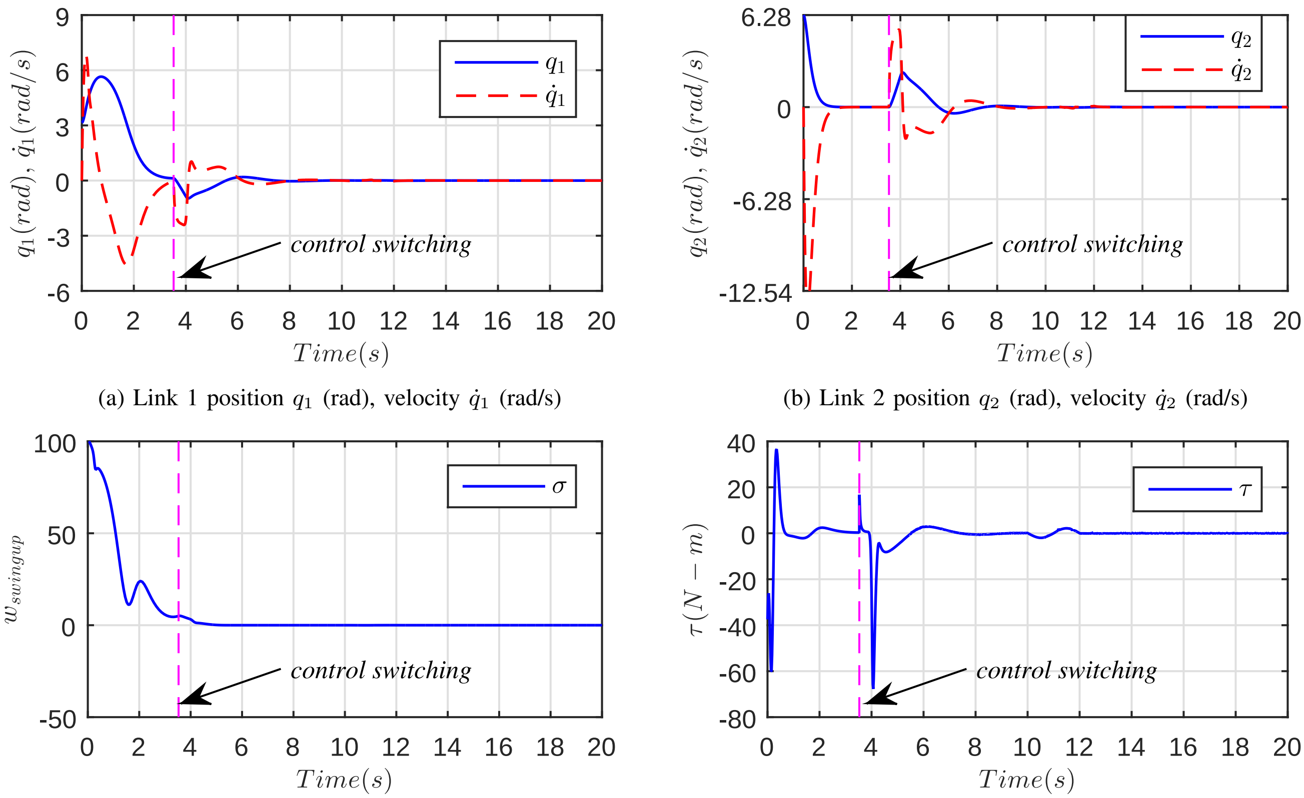


(c) Swingup Control $w_{swingup}$



(d) Control effort τ_{PFL2} (N-m)

Fig. 4: Acrobot - Closed loop response with Swingup Controller $w_{swingup}$ (39), $K_d = 12$, $K_p = 36$, initial condition $q(0) = [\pi, 0, 2\pi, 0]^T$, applied disturbance $d(t) = 2\sin(\pi t)$ from $t = 10$ (s) to $t = 12$ (s)



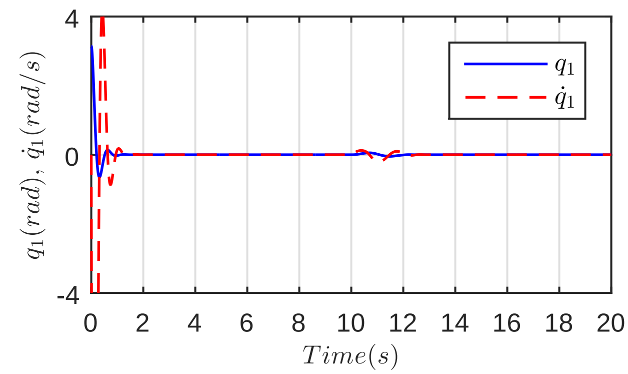
(a) Link 1 position q_1 (rad), velocity $\dot{q}_1$ (rad/s)

(b) Link 2 position q_2 (rad), velocity $\dot{q}_2$ (rad/s)

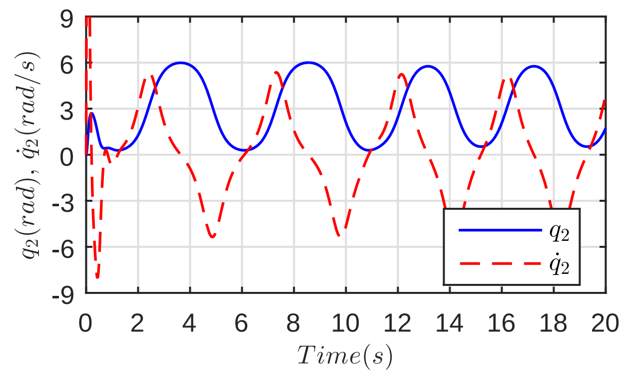
(c) Sliding surface σ

(d) Control effort τ (N-m)

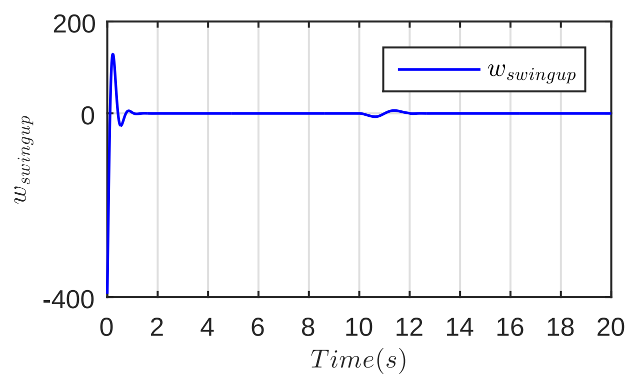
Fig. 5: Acrobot - Closed loop response with $w_{swingup}$ (39), $K_d = 12$, $K_p = 36$, and u (23) ($\rho = 3$, $K_1 = 75$, $K_2 = 50$), observer (24) ($\lambda_1 = 1$, $\lambda_2 = 3$), sliding parameters ($\alpha = 8$, $\beta = 16$), initial condition $q(0) = [\pi, 0, 2\pi, 0]^T$, applied disturbance $d(t) = 2\sin(\pi t)$ from $t = 10$ (s) to $t = 12$ (s)



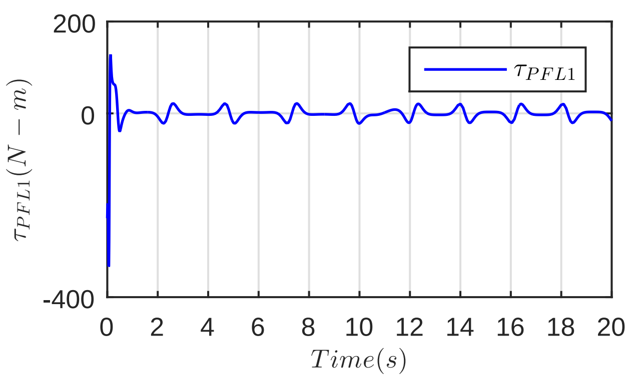
(a) Link 1 position q_1 (rad), velocity $\dot{q}_1$ (rad/s)



(b) Link 2 position q_2 (rad), velocity $\dot{q}_2$ (rad/s)

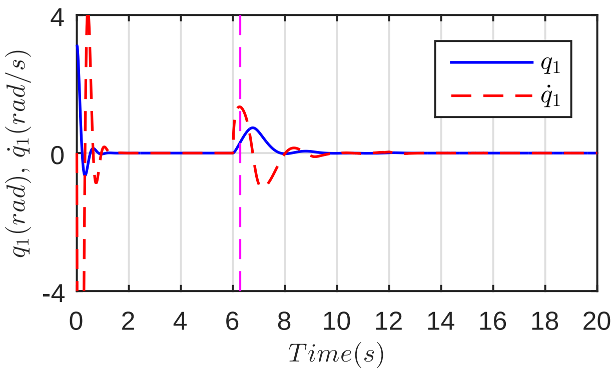


(c) Swingup Control $w_{swingup}$

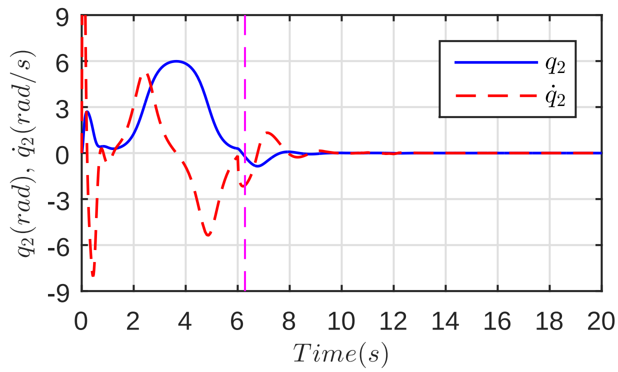


(d) Control effort τ_{PFL1} (N-m)

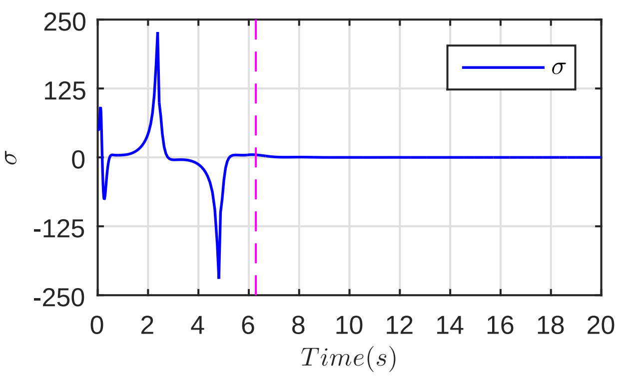
Fig. 6: Pendubot - Closed loop response with Swingup Controller $w_{swingup}$ (44), $K_d = 10$, $K_p = 125$, initial condition $q(0) = [\pi, 0, 0, 0]^T$, applied disturbance $d(t) = 5\sin(\pi t)$ from $t = 10$ (s) to $t = 12$ (s)



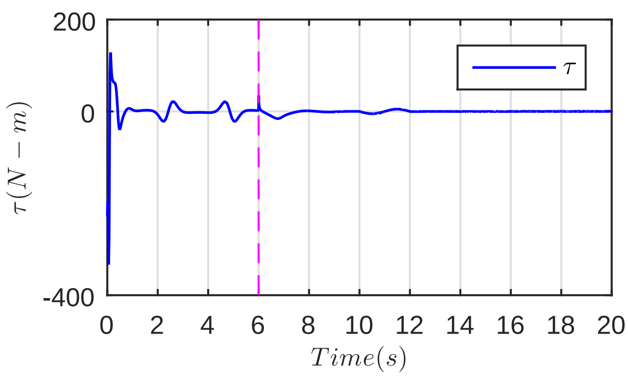
(a) Link 1 position q_1 (rad), velocity $\dot{q}_1$ (rad/s)



(b) Link 2 position q_2 (rad), velocity $\dot{q}_2$ (rad/s)



(c) Sliding surface σ



(d) Control effort τ (N-m)

Fig. 7: Pendubot - Closed loop response $w_{swingup}$ (44), $K_d = 10$, $K_p = 125$, and u (23) ($\rho = 3$, $K_1 = 125$, $K_2 = 100$), observer (24) ($\lambda_1 = 1$, $\lambda_2 = 3$), sliding parameters ($\alpha = 8$, $\beta = 16$), initial condition $q(0) = [\pi, 0, 0, 0]^T$, applied disturbance $d(t) = 5\sin(\pi t)$ from $t = 10$ (s) to $t = 12$ (s)